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On fundamental solutions, transition probabilities and fractional derivatives

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The aim of this note is to clarify the connection between different notions of fundamental solution and to outline the interplay between transitional probabilities of stochastic processes, evolution semigroups, evolution equations and their fundamental solutions. We discuss different notions of the fundamental solution for Lévy processes with infinitely smooth symbol and for stable subordinators. In the case of Lévy processes with infinitely smooth symbol we find the fundamental solution of the corresponding forward evolution equation and recover the Duhamel formula for the solution of the Cauchy problem for this equation. In the case of the $1/2$-stable subordinator, we find the transition density by solving an evolution equation with the (weak) Riemann — Liouville fractional derivative and show that the Weyl fractional derivative is the negative of the adjoint to the Riemann — Liuoville (weak) fractional derivative.

Keywords: evolution semigroups, fundamental solution, fractional derivative, subordinator

Introduction

In the series of papers [2, 3, 4, 5] a technique to construct evolution semigroups $(T_t)_{t\geq 0}$ generated by some operators L was developed. In the frame of the suggested technique the following fact was used: the identity

$$\int\limits_s^\infty T_{t-s}[\xi'(t) + L\xi(t)]\,dt = -\xi(s) \tag{1}$$

is true for each "test-function" $\xi\colon \mathbb{R} \to \mathrm{Dom}(L)$ and each $s \in \mathbb{R}$. Here $\mathrm{Dom}(L)$ is the domain of the generator L. The object $(T_t)_{t\geq 0}$, satisfying the identity (1) with a given operator L was called *fundamental solution* of $\partial_t + L$. It was shown in the paper [1, Th. 4.1] that this object $(T_t)_{t\geq 0}$ is indeed the semigroup generated by L and there are no other candidates except $\partial_t + L$ to fulfill (1) with the given $(T_t)_{t\geq 0}$. The technique of [2, 3, 4, 5] was used in [1] in particular to discuss evolution semigroups generated by additive perturbations of the $(1/2)$-stable subordinator, i.e. of the operator L equal to the Weyl fractional derivative of order $1/2$.

This note is supposed to be an addition to the discussion of [1]. The aim of this note is to clarify the connection between the notion of fundamental solution presented above and the traditional notion used in the Theory of Partial Differential Equations and in Functional Analysis (cf. [7])

and to outline the interplay between transitional probabilities of stochastic processes, evolution semigroups, evolution equations and their fundamental solutions. To make the picture as clear as possible we restrict the discussion to the case of Lévy processes with infinitely smooth symbol, e.g. with compactly supported Lévy measure, and to a class of subordinators. In the first case we find the (traditional) fundamental solution of the corresponding forward evolution equation, recover the Duhamel formula for the solution of the Cauchy problem for this equation and arrive at the identity (1).

In the case of the $(1/2)$-stable subordinator, the transition density is also obtained explicitly by solving an evolution equation with the (weak) Riemann–Liouville fractional derivative. To this end we show that the Weyl fractional derivative is the negative of the adjoint to the Riemann–Liuoville (weak) fractional derivative and again arrive at the identity (1).

1. Notations and definitions

Below we will use standard techniques of Fourier analysis and the Schwarz theory of distributions, for which we refer the reader to [7].

Let $\mathcal{D}(\mathbb{R}^d) := C_c^\infty(\mathbb{R}^d)$ be the space of test functions, i.e. infinitely smooth functions with compact supports. Let $\mathcal{D}'(\mathbb{R}^d)$ be the space of all generalized functions (distributions) on $\mathbb{R}^d$, i.e. the space dual to $\mathcal{D}(\mathbb{R}^d)$ taken with the standard topology. Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space of tempered functions. For each function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ let $\mathcal{F}[\varphi]$ be its Fourier transform defined as $\mathcal{F}[\varphi](p) = \int\limits_{\mathbb{R}^d} e^{-ip\cdot q}\varphi(q)\,dq$ and let $\mathcal{F}^{-1}$ be its inverse. Denote the space of tempered distributions as $\mathcal{S}'(\mathbb{R}^d)$, $\mathcal{S}'(\mathbb{R}^d) \subset \mathcal{D}'(\mathbb{R}^d)$. Denote the dual pairing between $\mathcal{D}'(\mathbb{R}^d)$ and $\mathcal{D}(\mathbb{R}^d)$ (and between $\mathcal{S}'(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$) as $\langle\cdot,\,\cdot\rangle$. Each locally absolutely integrable function $f \in L^1_{loc}(\mathbb{R}^d)$ corresponds to a regular generalized function (distribution) $f \in \mathcal{D}'(\mathbb{R}^d)$ acting by the formula $\langle f,\,\varphi\rangle := \int\limits_{\mathbb{R}^d} f(x)\varphi(x)dx$, $\varphi \in \mathcal{D}(\mathbb{R}^d)$. We use the same notation $\langle f,\,g\rangle$ for the integral $\int\limits_{\mathbb{R}^d} f(x)g(x)dx$ for all such functions f, g: $\mathbb{R}^d \to \mathbb{C}$ that the integral is well-defined. Any σ-finite Borel measure μ defines a distribution $\left\{\mu \,\middle|\, \langle\mu,\,\varphi\rangle = \int\limits_{\mathbb{R}^d}\varphi(x)\mu(dx)\right\}$, $\varphi \in \mathcal{D}(\mathbb{R}^d)$. The Dirac delta-function δ is a distribution corresponding to a Borel measure that assigns unit mass to the point $x = 0$, i.e. $\langle\delta,\,\varphi\rangle = \varphi(0)$. Note, that $\delta \in \mathcal{S}'(\mathbb{R}^d) \subset \mathcal{D}'(\mathbb{R}^d)$.

Fundamental solution. Let $\mathcal{L}$ be a linear operator on $\mathcal{D}'(\mathbb{R}^d)$ (resp. on $\mathcal{S}'(\mathbb{R}^d)$). A *fundamental solution* of the operator $\mathcal{L}$ is any function $\mathcal{E} \in \mathcal{D}'(\mathbb{R}^d)$ (resp. $\mathcal{E} \in \mathcal{S}'(\mathbb{R}^d)$) solving in $\mathcal{D}'(\mathbb{R}^d)$ (resp. in $\mathcal{S}'(\mathbb{R}^d)$) the equation

$$\mathcal{L}\mathcal{E} = \delta,$$

i.e. for each test function $\varphi \in \mathcal{D}'(\mathbb{R}^d)$ (resp. $\varphi \in \mathcal{S}'(\mathbb{R}^d)$) holds the identity

$$\langle\mathcal{L}\mathcal{E},\,\varphi\rangle = \varphi(0).$$

Pseudo-differential operators on the space of tempered distributions. For each $f \in \mathcal{S}'(\mathbb{R}^d)$ its Fourier transform $\mathcal{F}[f]$ is defined by $\langle\mathcal{F}[f],\,\varphi\rangle = \langle f,\,\mathcal{F}[\varphi]\rangle$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$. The operation of multiplication on an infinitely differentiable function $\psi : \mathbb{R}^d \to \mathbb{C}$, which grows at infinity with all its derivatives at most as a polynomial, is also well defined in $\mathcal{S}'(\mathbb{R}^d)$ by the formula $\langle \psi f,\, \varphi \rangle := \langle f,\, \psi\varphi \rangle$. In the sequel $\psi \in C^\infty(\mathbb{R}^d)$ is a continuous negative definite function, i.e. ψ is given by the Levy — Khinchine formula

$$\psi(p) = a(q) + i\ell(q) \cdot p + p \cdot Q(q)p + \int\limits_{y \neq 0} \left(1 - e^{ip\cdot y} + \frac{ip \cdot y}{1 + |y|^2}\right) N(dy), \quad p \in \mathbb{R}^d,$$

where, for each fixed $q \in \mathbb{R}^d$, $\ell(q) \in \mathbb{R}^d$, $Q(q)$ is a positive semidefinite symmetric matrix and $\nu(dy)$ is a measure kernel on $\mathbb{R}^d \setminus \{0\}$ such that

$$\int\limits_{y \neq 0} \frac{|y|^2}{1 + |y|^2} N(dy) < \infty.$$

Note, that ψ grows at infinity with all its derivatives not faster than a polynomial [6, Th. 3.7.13]. A pseudo-differential operator with the symbol ψ is defined on $\mathcal{S}(\mathbb{R}^d)$ as a composition $\mathcal{F}^{-1}\psi\mathcal{F}$. Note also that $\mathcal{F}^{-1}\psi(\xi)\mathcal{F} = \mathcal{F}\psi(-\xi)\mathcal{F}^{-1}$. The extension of this operator to the space $\mathcal{S}'(\mathbb{R}^d)$ is defined by

$$\langle \mathcal{F}^{-1}\psi\mathcal{F} f,\, \varphi \rangle := \langle f,\, \mathcal{F}[\psi(\mathcal{F}^{-1}\varphi)] \rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}^d), \;\; f \in \mathcal{S}'(\mathbb{R}^d).$$

Convolution of distributions. The operation of convolution $*$ is defined for several distributions $f, g \in \mathcal{D}'(\mathbb{R}^d)$ by the formula

$$\langle f * g,\, \varphi \rangle := \ll f(x)g(y),\, \varphi(x + y) \gg.$$

Here $\ll \cdot,\, \cdot \gg$ is the dual pairing between $\mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^d)$ and $\mathcal{D}(\mathbb{R}^d \times \mathbb{R}^d)$, the distribution $f(x)g(y) \in \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^d)$ is a direct product of f and g and it is supposed that the distribution $f(x)g(y)$ is correctly defined on all functions $\varphi(x+y)$, $\varphi \in \mathcal{D}(\mathbb{R}^d)$, although $\varphi(x+y)$ do not lie in $\mathcal{D}(\mathbb{R}^d \times \mathbb{R}^d)$ any more. In the case f, g are regular distributions, one has

$$\ll f(x)g(y),\, \varphi(x + y) \gg := \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} f(x)g(y)\varphi(x + y)\, dxdy.$$

Fundamental solution of a convolution operator. Let $\mathcal{D}'_+ \subset \mathcal{D}'(\mathbb{R})$ be the set of all generalized functions, whose supports are in $[0, +\infty)$. The set $\mathcal{D}'_+$ is a convolution algebra, i.e. associative and commutative algebra with the operation of convolution $*$, and the unit is the Dirac delta-function δ. Each element $f \in \mathcal{D}'_+$ defines a convolution operator $L_f \equiv f*$ such that for all $u \in \mathcal{D}'_+$ we have $L_f(u) = f * u$. A fundamental solution of a convolution operator L_f (if exists) is a function $\mathcal{E}_f \in \mathcal{D}'_+$ such that $L_f(\mathcal{E}_f) \equiv f * \mathcal{E}_f = \delta$. Therefore, the fundamental solution $\mathcal{E}_f$ is an inverse element to f in the convolution algebra $\mathcal{D}'_+$.

The (weak) Riemann — Liouville fractional derivative. Let $\beta \in \mathbb{R}$. Consider a distribution (i.e. a generalized function) $f_\beta \in \mathcal{D}'_+$ defined by the formula

$$f_\beta = \begin{cases} \dfrac{\eta(x)}{\Gamma(\beta)} x^{\beta - 1}, & x \in \mathbb{R}, \;\; \beta > 0; \\ f^{(N)}_{\beta + N}, & \beta + N > 0, \;\; \beta \leq 0. \end{cases} \tag{2}$$

Here Γ is the Euler gamma-function, η is the Heaviside function and $f^{(N)}$ is the Nth derivative of the generalized function f. Therefore, $f_1 = \eta$, $f_0 = f_1' = \delta$ and for each $\beta, \gamma \in \mathbb{R}$ we have $f_\beta * f_\gamma = f_{\beta+\gamma}$. Hence for each $\beta \in \mathbb{R}$ the fundamental solution $\mathcal{E}_{f_\beta}$ of a convolution operator L_{f_β} exists and $\mathcal{E}_{f_\beta} = f_{-\beta}$. For $\beta = -n$, $n \in \mathbb{N}$, we have $f_{-n} = \delta^{(n)}$, i.e. $L_{f_{-n}}(u) = f_{-n}u = u^{(n)}$ for all $u \in \mathcal{D}'_+$. Moreover, for $\beta = n$, $n \in \mathbb{N}$ we have $f_n = f_1 * \cdots * f_1$ and $L_{f_n}(u) = \eta * \cdots * \eta * u$ is a n-fold antiderivative of the generalized function u. For all $u \in \mathcal{D}'_+$ we call $L_{f_\beta}u$ the (weak) Riemann — Liouville fractional derivative of u when $\beta < 0$, and the Riemann — Liouville fractional integral of u when $\beta > 0$. For the (weak) Riemann — Liouville derivative of the order $\nu > 0$ we will also use the notation ∂_x^ν, i.e. $\partial_x^\nu u(x) := L_{f_{-\nu}}(u)(x) \equiv f_{-\nu} * u(x)$.

Laplace transform of distributions. For each $a \geq 0$ define $\mathcal{D}'_+(a)$ as a set of such functions f from $\mathcal{D}'_+$ that $f(x)e^{-sx} \in \mathcal{S}'_+ \equiv \mathcal{D}'_+ \cap \mathcal{S}'(\mathbb{R})$ for all $s > a$. For each $a \geq 0$ the set $\mathcal{D}'_+(a)$ is called the set of originals with the growth rate up to a. Then $\mathcal{S}'_+ \subset \mathcal{D}'_+(0) \subset \mathcal{D}'_+(a_1) \subset \mathcal{D}'_+(a_2)$ for all $0 \leq a_1 \leq a_2$. Note that $\mathcal{S}'_+$ and $\mathcal{D}'_+(a)$ are convolution subalgebras of $\mathcal{D}'_+$ for all $a \geq 0$.

Let $f \in \mathcal{D}'_+(a)$. For arbitrary fixed $s > a$ define the Laplace transform $\mathfrak{L}[f]$ of f by the formula

$$\mathfrak{L}[f](p) = \langle f(x)e^{-sx},\ \eta(x)e^{-(p-s)x}\rangle,$$

where $\langle\cdot,\ \cdot\rangle$ is the dual pairing between the space of generalized functions $\mathcal{S}'_+$ and the space of test functions $\mathcal{S}_+ = \{\varphi\colon \mathbb{R} \to \mathbb{C} \mid \eta(x)\varphi(x) = \eta(x)\varphi(x) \text{ for some } \varphi \in \mathcal{S}(\mathbb{R})\}$.

Note that for each $\beta \in \mathbb{R}$ we have $f_\beta \in \mathcal{D}'_+(a)$ for all $a > 0$ and $\mathfrak{L}[f_\beta] = p^{-\beta}$. Moreover, the Laplace transform $\mathfrak{L}$ transforms a convolution into a product: $\mathfrak{L}[f*g] = \mathfrak{L}[f]\mathfrak{L}[g]$ for $f,\ g \in \mathcal{D}'_+(a)$. Therefore, the Laplace transform of the (weak) Riemann — Liouville fractional derivative (of the order $\nu > 0$) of a function $u \in \mathcal{D}'_+(a)$ has the form

$$\mathfrak{L}[\partial_x^\nu u] = \mathfrak{L}[f_{-\nu} * u] = p^\nu \mathfrak{L}[u].$$

2. Fundamental solutions, evolution semigroups and transition probabilities

Let $\{X_t\}_{t\geq 0}$ be a Lévy process on $\mathbb{R}^d$. Then the distribution $\mu := P_{X_1}$ of a random variable X_1 is infinitely divisible and defines a convolution semigroup $\{\mu^t\}_{t\geq 0}$ on $\mathbb{R}^d$. The transition function $P_t(x, B)$ of the Lévy process $\{X_t\}_{t\geq 0}$ is then defined by

$$P_t(x, B) := \mu^t(B - x), \quad t \geq 0,\ \ x \in \mathbb{R}^d,\ \ B \in \mathcal{B}(\mathbb{R}^d).$$

Using the convolution semigroup $\{\mu^t\}_{t\geq 0}$, one can construct two evolution semigroups on the space $C_\infty(\mathbb{R}^d)$ of continuous vanishing at infinity functions: the backward semigroup $(T_t)_{t\geq 0}$ and the forward semigroup $(T_t^*)_{t\geq 0}$. These semigroups are defined for $f \in C_\infty(\mathbb{R}^d)$ as follows:

$$T_t f(x) := \mathbb{E}[f(x + X_t)] = \int\limits_{\mathbb{R}^d} f(x+y)\mu^t(dy) = \int\limits_{\mathbb{R}^d} f(y)P_t(x, dy);$$

$$T_t^* f(x) := \mathbb{E}[f(x - X_t)] = \int\limits_{\mathbb{R}^d} f(x-y)\mu^t(dy) = f * \mu^t(x).$$

Proposition 1. The semigroups T_t and T_t^* can be naturally extended to the space $L^2(\mathbb{R}^d)$ and these extensions are adjoint, i.e. for all $f,\ g \in L^2(\mathbb{R}^d)$ one has

$$\langle T_t f,\ g\rangle = \langle f,\ T_t^* g\rangle.$$

Moreover, operators T_t^* are pseudo-differential operators with symbol $e^{-t\psi}$ and operators T_t are pseudo-differential operators with symbol $e^{-t\psi(-\cdot)}$ for some continuous negative definite function ψ.

Proof. Indeed, for all $f,\ g \in C_\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$

$$\langle T_t f,\ g\rangle = \int\limits_{\mathbb{R}^d} T_t f(x) g(x)\, dx = \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} f(x+y)\mu^t(dy) g(x)\, dx = \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} f(z) g(z-y)\mu^t(dy)\, dz = \int\limits_{\mathbb{R}^d} f(z) T_t^* g(z)\, dz = \langle f,\ T_t^* g\rangle.$$

By the Bochner theorem there exist a unique continuous negative definite function ψ such that $\mathcal{F}[\mu^t] = e^{-t\psi}$. Then, using the properties of convolution and Fourier transform, one has for $f \in \mathcal{S}(\mathbb{R}^d)$

$$T_t^* f = f * \mu^t = \mathcal{F}^{-1}\Big[\mathcal{F}[f * \mu^t]\Big] = \mathcal{F}^{-1}\Big[\mathcal{F}[f]\mathcal{F}[\mu^t]\Big] = \mathcal{F}^{-1}\Big[e^{-t\psi}\mathcal{F}[f]\Big],$$

i.e. the semigroup T_t^* is a family of pseudo-differential operators with the symbol $e^{-t\psi}$. Let now L be the generator of $(T_t)_{t\geq 0}$ and L^* be the generator of $(T_t^*)_{t\geq 0}$. Hence the generator L^* is also a pseudo-differential operator with the symbol $-\psi$. Further, one has for $f, g \in \mathcal{S}(\mathbb{R}^d)$

$$\langle f,\ T_t^* g\rangle = \langle f,\ \mathcal{F}^{-1}\Big[e^{-t\psi}\mathcal{F}[g]\Big]\rangle = \langle \mathcal{F}\Big[e^{-t\psi}\mathcal{F}^{-1}[f]\Big],\ g\rangle = \langle \mathcal{F}^{-1}\Big[e^{-t\psi(-\cdot)}\mathcal{F}[f]\Big],\ g\rangle = \langle T_t f,\ g\rangle,$$

i.e. the semigroup T_t is a family of pseudo-differential operators with the symbol $e^{-t\psi(-\cdot)}$. And hence the generator L is a pseudo-differential operator with the symbol $-\psi(-\cdot)$. Moreover, for all $f,\ g \in \mathcal{S}(\mathbb{R}^d)$ one has $\langle Lf,\ g\rangle = \langle f,\ L^* g\rangle$. The proposition is proved.

Since $\psi \in C^\infty(\mathbb{R}^d)$ grows at infinity with all its derivatives not faster than a polynomial then for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ the functions $\mathcal{F}^{-1}\Big[-\psi\mathcal{F}[\varphi]\Big]$ and $\mathcal{F}^{-1}\Big[-\psi(-\cdot)\mathcal{F}[\varphi]\Big]$ are well defined and belong again to $\mathcal{S}(\mathbb{R}^d)$. Hence one can define the operators L and L^* on the space $\mathcal{S}'(\mathbb{R}^d)$ by the formulas $Lf := \mathcal{F}^{-1}\Big[-\psi(-\cdot)\mathcal{F}[f]\Big]$ and $L^* f := \mathcal{F}^{-1}\Big[-\psi\mathcal{F}[f]\Big]$ respectively, i.e. for each $f \in \mathcal{S}'(\mathbb{R}^d)$ and each $\varphi \in \mathcal{S}(\mathbb{R}^d)$ one has:

$$\langle L^* f,\ \varphi\rangle := \Big\langle \mathcal{F}^{-1}\Big[-\psi\mathcal{F}[f]\Big],\ \varphi\Big\rangle = \Big\langle f,\ \mathcal{F}\Big[-\psi\mathcal{F}^{-1}[\varphi]\Big]\Big\rangle = \langle f,\ L\varphi\rangle$$

and vice versa $\langle Lf,\ \varphi\rangle := \langle f,\ L^*\varphi\rangle$.

Connections between semigroups, evolution equations and their fundamental solutions. Consider now the Cauchy problem in $\mathbb{R}^d$

$$\begin{aligned} &\frac{\partial f}{\partial t}(t,x) = L^* f(t,x), \\ &f(0,x) = f_0(x). \end{aligned} \tag{3}$$

Here $f_0 \in \mathcal{S}(\mathbb{R}^d)$ and the problem is well-posed in $L^2(\mathbb{R}^d)$. The theory of evolution semigroups provides the solution of (3) in the form

$$f(t,x) = T_t^* f_0(x).$$

This classical Cauchy problem (3) can be also transformed into the generalized one in a standard way (see [7]): let $f(t,x)$ be a solution of (3). For each $t < 0$ and all $x \in \mathbb{R}^d$ define $f(t,x) := 0$ and consider the function f as an element of $\mathcal{S}'(\mathbb{R} \times \mathbb{R}^d)$. Then the weak derivative $\partial_t f$ of f with respect to the variable t is calculated as follows:

$$\partial_t f(t,x) = \frac{\partial f}{\partial t}(t,x) + f_0(x)\delta(t),$$

here $\frac{\partial f}{\partial t}$ is the classical derivative and f_0 is the initial data of the Cauchy problem (3). And hence the solution of the classical Cauchy problem (3) solves the equation

$$\partial_t f(t,x) - L^* f(t,x) = f_0(x)\delta(t) \tag{4}$$

in $\mathcal{S}'(\mathbb{R} \times \mathbb{R}^d)$.

Assume that a fundamental solution $\mathcal{E}$ of the operator $\partial_t - L^*$ in $\mathcal{S}'(\mathbb{R} \times \mathbb{R}^d)$ exists. If L^* is a local operator than our assumption is true (see [7]) and the solution of (4) is given then by the Duhamel formula

$$f(t,x) = [\mathcal{E}(t,x)] * [f_0(x)\delta(t)] = [\mathcal{E}(t,\cdot) * f_0](x). \tag{5}$$

Proposition 2. The Duhamel formula (5) is also true in the case when L^* is the generator of the Lévy process X_t, whose symbol ψ is of class $C^\infty(\mathbb{R}^d)$.

Proof. Let us solve the equation $\partial_t \mathcal{E} - L^* \mathcal{E} = \delta$ in $\mathcal{S}'(\mathbb{R} \times \mathbb{R}^d)$. Apply the Fourier transform with respect to the variable x. Let $\mathcal{F}[\mathcal{E}(t,\cdot)](y) = \widehat{\mathcal{E}}(t,y)$. Then we have $\mathcal{F}[\partial_t \mathcal{E}(t,\cdot)](y) = \partial_t \widehat{\mathcal{E}}(t,y)$, $\mathcal{F}[L^* \mathcal{E}(t,\cdot)](y) = -\psi(y)[\widehat{\mathcal{E}}(t,y)]$ and $\mathcal{F}[\delta(t,\cdot)](y) = \delta(t)$. Therefore, we obtain

$$\partial_t \widehat{\mathcal{E}}(t,y) + \psi(y)\widehat{\mathcal{E}}(t,y) = \delta(t),$$

i.e. $\widehat{\mathcal{E}}(\cdot,y)$ is the fundamental solution of an ordinary (with respect to the variable t, y is a parameter) differential operator $\widehat{L}_y = \partial_t + \psi(y)$. Let us find the fundamental solution of $\widehat{L}_y$ using the Laplace transform with respect to the variable t. Let $\mathfrak{L}[\widehat{\mathcal{E}}(t,y)](s) = \widehat{\mathbf{E}}(s,y)$. Then $\mathfrak{L}[\partial_t \widehat{\mathcal{E}}(t,y)](s) = s\widehat{\mathbf{E}}(s,y)$ and $\mathfrak{L}[\delta(t)](s) = 1$. Therefore, we get

$$s\widehat{\mathbf{E}}(s,y) + \psi(y)\widehat{\mathbf{E}}(s,y) = 1,$$

i.e.

$$\widehat{\mathbf{E}}(s,y) = \frac{1}{s + \psi(y)}.$$

Hence $\widehat{\mathcal{E}}(t,y) = e^{-t\psi(y)}\eta(t)$ with the Heaviside function η. And, therefore,

$$\mathcal{E}(t,x) = \eta(t)\mathcal{F}^{-1}[e^{-t\psi}](x).$$

Since $\psi(\cdot)$ is a continuous negative definite function, the function $e^{-t\psi(\cdot)}$ is positive definite and its inverse Fourier transform is the measure μ^t, i.e. $\mathcal{E}(t,\cdot) = \eta(t)\mu^t$. Therefore, for all $t \geq 0$ and $x \in \mathbb{R}^d$

$$[\mathcal{E}(t,\cdot) * f_0](x) = \int\limits_{\mathbb{R}^d} f_0(x-y)\mu^t(dy) = T_t^* f_0(x) = f(t,x),$$

i.e. the Duhamel formula (5) indeed solves the Cauchy problem (3).

Connection between different notions of the fundamental solution. The identity $\partial_t\mathcal{E} - L^*\mathcal{E} = \delta$ in $\mathcal{S}'(\mathbb{R}\times\mathbb{R}^d)$ means that for each $\varphi \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^d)$ one has

$$\langle \partial_t\mathcal{E} - L^*\mathcal{E},\ \varphi\rangle = \langle \delta,\ \varphi\rangle = \varphi(0,0).$$

Let us fix $s \in \mathbb{R}$, $x \in \mathbb{R}^d$. With a linear change of variables $t \mapsto t-s$, $y \mapsto y-x$ one can show that the generalized function (distribution) $\mathcal{E}_{s,x} \in \mathcal{S}'(\mathbb{R}\times\mathbb{R}^d)$,

$$\mathcal{E}_{s,x}(t,dy) := \mathcal{E}(t-s, dy-x),$$

solves in $\mathcal{S}'(\mathbb{R}\times\mathbb{R}^d)$ the equation $\partial_t\mathcal{E}_{s,x} - L^*\mathcal{E}_{s,x} = \delta_{s,x}$ with the shifted Dirac delta-function $\delta_{s,x}$ such that $\langle \delta_s x, \varphi, =\rangle \varphi(s,x)$. The function $\mathcal{E}_{s,x}$ is usually called a fundamental solution with singularity at (s,x). Therefore, for each $\varphi \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^d)$ one has:

$$\begin{aligned}\varphi(s,x) = \langle \partial_t\mathcal{E}_s x - L^*\mathcal{E}_{s,x}, \varphi, =\rangle\langle \mathcal{E}_s x, -\partial_t\varphi - L\varphi, =\rangle \\ - \int\limits_{\mathbb{R}} dt \int\limits_{\mathbb{R}^d} [\partial_t\varphi(t,y) + L\varphi(t,y)]\eta(t-s)\mu^{t-s}(dy-x) = \\ - \int\limits_s^{+\infty}\int\limits_{\mathbb{R}^d} [\partial_t\varphi(t,y) + L\varphi(t,y)]P_{t-s}(x,dy)\,dt.\end{aligned}$$

Therefore, the identity (1) (with $\xi(t) := \varphi(t,\cdot)$) is recovered for this particular case in the form:

$$\int\limits_s^{+\infty}\int\limits_{\mathbb{R}^d} [\partial_t\varphi(t,y) + L\varphi(t,y)]P_{t-s}(x,dy)\,dt = -\varphi(s,x), \quad \varphi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}\times\mathbb{R}^d).$$

3. Fractional derivatives and fundamental solutions

Define now the Weyl fractional derivative $(\partial_x^\nu)^*$ of order $\nu \in (0,1)$ for all test functions $\varphi \in \mathcal{D}(\mathbb{R})$ by the formula

$$(\partial_x^\nu)^*(\varphi)(x) = \frac{1}{\Gamma(1-\nu)}\int\limits_x^{+\infty}(y-x)^{-\nu}\varphi'(y)\,dy.$$

Proposition 3. The operator $-(\partial_x^\nu)^*$ is adjoint to the weak Riemann — Liouville derivative (∂_x^ν) in the following sense: for all $\varphi \in \mathcal{D}(\mathbb{R})$ and $g \in \mathcal{D}'_+(\mathbb{R}) \subset \mathcal{D}'(\mathbb{R})$ one has

$$\langle \partial_x^\nu g,\ \varphi\rangle = -\langle g,\ (\partial_x^\nu)^*\varphi\rangle.$$

Proof. Indeed, using the rule of differentiation of a convolution one has

$$\langle \partial_x^\nu g,\, \varphi\rangle = \langle f_{-\nu} * g,\, \varphi\rangle = \langle f'_{1-\nu} * g,\, \varphi\rangle = \langle (f_{1-\nu} * g)',\, \varphi\rangle = -\langle g * f_{1-\nu},\, \varphi'\rangle =$$
$$- \ll f_{1-\nu}(y)g(x),\, \varphi'(x+y)\gg = \Big\langle g(x),\, \langle f_{1-\nu},\, \varphi'(x+\cdot)\rangle\Big\rangle = -\langle g,\, (\partial_x^\nu)^*\varphi\rangle.$$

Here we used the fact that, since $f_{1-\nu} \in L^1_{\mathrm{loc}}(\mathbb{R})$ is a regular distribution, one has

$$\langle f_{1-\nu},\, \varphi'(x+\cdot)\rangle = \int\limits_{\mathbb{R}} f_{1-\nu}(y)\varphi'(x+y)\,dy = \frac{1}{\Gamma(1-\nu)} \int\limits_x^{+\infty} (z-x)^{-\nu}\varphi'(z)\,dz = (\partial_x^\nu)^*\varphi(x).$$

Connection between different notions of the fundamental solution. Let $\mathcal{D}'_+(\mathbb{R}^2)$ be the set of generalized functions from $\mathcal{D}'(\mathbb{R}^2)$ having supports in the quadrant $\{(t,x) \in \mathbb{R}^2 \mid t \geq 0,\ x \geq 0\}$. Let $\nu \in (0,1)$. Let there exists a function $\mathcal{E}^\nu(t,x) \in \mathcal{D}'_+(\mathbb{R}^2)$ which is a fundamental solution of the operator $L_\nu := \partial_t + \partial_x^\nu$ with the weak Riemann — Liouville fractional derivative ∂_x^ν, i.e. $\mathcal{E}^\nu$ solves in $\mathcal{D}'(\mathbb{R}^2)$ the equation $L_\nu\mathcal{E}^\nu = \delta$. As in the previous Subsection fix $s \in \mathbb{R}$ and $y \in \mathbb{R}$. With a linear change of variables $t \mapsto t-s$, $x \mapsto x-y$ one can show that the generalized function (distribution) $\mathcal{E}^\nu_{s,y} \in \mathcal{D}'(\mathbb{R}^2)$,

$$\mathcal{E}^\nu_{s,y}(t,x) := \mathcal{E}^\nu(t-s, x-y),$$

i.e.

$$\langle \mathcal{E}^\nu_s y,\, \varphi(\cdot,\cdot)\rangle = \langle \mathcal{E}^\nu,\, \varphi(\cdot+s, \cdot+y)\rangle,$$

solves in $\mathcal{D}'(\mathbb{R}^2)$ the equation $\partial_t\mathcal{E}^\nu_{s,y} + \partial_x^\nu\mathcal{E}_{s,y} = \delta_{s,y}$ with the shifted Dirac delta-function $\delta_{s,y}$. Hence

$$\varphi(s,y) = \langle \delta_s y,\, \varphi\rangle = \langle \partial_t\mathcal{E}^\nu_s y + \partial_x^\nu\mathcal{E}_{s,y},\, \varphi\rangle = -\langle \mathcal{E}^\nu_s y,\, \partial_t\varphi + (\partial_x^\nu)^*\varphi\rangle.$$

Assume now that $\mathcal{E}^\nu$ is a regular distribution (it is so, e.g., for $\nu = 1/2$ due to Lemma below). Then for each test function $\varphi(t,x) \in \mathcal{D}(\mathbb{R}^2)$ the function $\mathcal{E}^\nu(t,x)$ satisfies the identity

$$\int\limits_s^\infty \int\limits_y^\infty \mathcal{E}^\nu(t-s, x-y)[\partial_t\varphi(t,x) + (\partial_x^\nu)^*\varphi(t,x)]\,dxdt = -\varphi(s,y), \quad s,\, y \in \mathbb{R}. \tag{6}$$

This formula once again recovers the identity (1) for the case of ν-stable subordinators.

Lemma 1. A fundamental solution of the operator $L_\nu = \partial_t + \partial_x^\nu$ for $\nu = 1/2$ is the function

$$\mathcal{E}^{1/2}(t,x) = \frac{t}{2\sqrt{\pi x^3}} e^{-\frac{t^2}{4x}} \eta(t)\eta(x).$$

Proof. Let us solve the equation $L_\nu\mathcal{E}^\nu = \delta$ in $\mathcal{D}'(\mathcal{R}^2)$. This equation is equivalent to $\partial_t\mathcal{E}^\nu(t,x) + \partial_x^\nu\mathcal{E}_\nu(t,x) = \delta(t)\delta(x)$. Applying Laplace transform with respect to the variable x to the generalized functions in both sides of the last equation and denoting $E_\nu(t,p) = \mathfrak{L}[\mathcal{E}^\nu(t,\cdot)](p)$ we get

$$\partial_t E_\nu(t,p) + p^\nu E_\nu(t,p) = \delta(t).$$

Applying Laplace transform with respect to the variable t to the generalized functions of the variable t in both sides of the last equation and denoting $G_\nu(s,p) = \mathfrak{L}[E_\nu(\cdot,p)](s)$ we get

$$sG_\nu(s,p) + p^\nu G_\nu(s,p) = 1.$$

Therefore, $G_\nu(s,p) = \dfrac{1}{s+p^\nu}$. Hence using the tables of Laplace transforms, one gets

$$E_\nu(t,p) = e^{-tp^\nu}\eta(t),$$

i.e. $E_{1/2}(t,p) = e^{-t\sqrt{p}}\eta(t)$. Once again, using the tables of Laplace transforms one gets that

$$\mathcal{E}_{1/2}(t,x) = \frac{t}{2\sqrt{\pi x^3}}e^{-\frac{t^2}{4x}}\eta(t)\eta(x).$$

Since the function $\mathcal{E}_{1/2}(t,x)$ is a regular distribution and belongs to the set $\mathcal{D}'_+(\mathbb{R}^2)$ then the equality (6) for $\nu = 1/2$ and $\mathcal{E}_{1/2}(t,x)$ holds.

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О фундаментальных решениях, переходных вероятностях и дробных производных

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Ключевые слова: эволюционные полугруппы; фундаментальное решение; дробная производная; субординатор

В серии недавних работ К. Богдана, В. Хансена, Т. Якубовского, К. Шипковского, С. Сидора [2, 3, 4, 5] была развита техника построения эволюционных полугрупп, порожденных некоторыми заданными операторами. В рамках предложенной техники использовался тот факт, что полугруппа является ядром отрицательного левого обратного оператора к сумме временнóй производной и генератора этой полугруппы. И это ядро называлось (слабым) фундаментальным решением данной суммы. Этот факт был доказан в статье К. Богдана, Я.А. Бутко и К. Шипковского [1], а именно, было показано, что полугруппа, порожденная заданным оператором, действительно является (слабым) фундаментальным решением упомянутой выше суммы, и только ее. В указанной статье техника К. Богдана и его соавторов была использована для обсуждения эволюционных полугрупп, порожденных аддитивными возмущениями $(1/2)$-устойчивого субординатора, т.е. оператора дробной производной Вейля порядка $1/2$.

Настоящая работа служит дополнением к вышеназванной статье. Целью является прояснение взаимосвязи между (слабым) фундаментальным решением, описанным выше, и традиционным понятием фундаментального решения, используемым в теории уравнений с частными производными и в функциональном анализе. В работе также обрисованы соотношения между переходными вероятностями случайных процессов, эволюционными полугруппами, эволюционными уравнениями и их фундаментальными решениями. Различные понятия фундаментального решения обсуждаются для процессов Леви с бесконечно гладким символом и для устойчивых субординаторов. В случае процессов Леви с бесконечно гладким символом найдено фундаментальное решение соответствующего прямого эволюционного уравнения и установлена формула Дюамеля для решения соответствующей задачи

Коши. В случае $(1/2)$-устойчивого субординатора найдено фундаментальное решение (переходная вероятность) путем решения эволюционного уравнения со (слабой) дробной производной Римана — Лиувилля и показано, что дробная производная Вейля является минус сопряженным оператором к (слабой) производной Римана — Лиувилля.

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